# Quark and gluon confinement in Coulomb gauge

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The Yang-Mills Schrödinger equation is variationally solved in Coulomb gauge for the vacuum sector using a trial wave functional, which is strongly peaked at the Gribov horizon. We find the absence of gluons in the infrared and also a confining quark potential.

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*1. Introduction.* One of the most challenging problem in particle physics is to explain the confinement of quarks and gluons in QCD. Several confinement mechanisms have been proposed in the past, the most prominent of which are perhaps the condensation of magnetic monopoles (dual Meissner effect) and center vortex condensation (for a recent review see ref.[1]). Evidence for both mechanisms has been found in lattice calculations. However, the center vortex picture is perhaps more advantageous in the sense, that these vortices can, in principle, be defined in a gauge invariant way [2], while magnetic monopoles arise as gauge artifacts after Abelian projection and are merely manisfestations of topological defects of the underlying gauge fields [3]. Yet another confinement mechanism was proposed by Gribov [4] and further elaborated in ref.[5]. This mechanism, which is based on the infrared dominance of the field configurations near the Gribov horizon in Coulomb gauge, is compatible with the magnetic monopole and center vortex pictures of confinement, given the fact, that these field configurations lie on the Gribov horizon [6].

In this letter we will explore Gribov's confinement mechanism by studying the vacuum sector of $SU(2)$ Yang-Mills theory in Coulomb gauge in the Hamilton approach using the variational principle. With an appropriate, physically motivated ansatz for the Yang-Mills wave functional, which accounts for the dominance of the field configurations on the Gribov horizon, we will be able to describe simultanously quark and gluon confinement.

In previous studies of the Yang-Mills Schrödinger equation in Coulomb gauge [7, 8], a different trial wave function was used and also the non-trivial metric of the orbit space, induced by the Faddeev-Popov determinant was not fully included. In ref.[7] the curvature of orbit space was completely neglected, while in ref.[8] its contribution to the gluon self-energy was partially omitted in the gap equation. We will find, however, that the proper inclusion of the Faddeev-Popov determinant is absolutely necessary to produce simultaneously quark and gluon confinement.

*2. Yang-Mills theory in Coulomb gauge.* The Hamilton approach to gauge theory is based on the Weyl gauge $A_0 = 0$, in which the dynamical degrees of freedom are the spatial components of the gauge field, $\vec{A}(x)$. This gauge leaves still invariance under spatial gauge transformations. The latter can be fixed by implementing the Coulomb gauge $\vec{\partial}\vec{A} = 0$. In the Coulomb gauge the unphysical longitudinal degrees of freedom of $\vec{A}(x)$ can be completely eliminated by explicitly resolving Gauß' law $\hat{D}\Pi|\Psi\rangle = \rho_{ext}|\Psi\rangle$, resulting in the following Hamiltonian of the physical transversal degrees of freedom $A_i^\perp$ ($\partial_i A_i^\perp = 0$)

$$\begin{aligned} H \;=\; & \frac{1}{2}\int d^3x \left[ g^2 \mathcal{J}^{-1}[A^\perp]\Pi_i^a(x)\mathcal{J}[A^\perp]\Pi_i^a(x) + \frac{1}{g^2}(B_i^a(x))^2 \right] \\ & + \frac{g^2}{2}\int d^3x \int d^3x' \mathcal{J}^{-1}[A^\perp]\rho^a(x)\langle a,x|(-\vec{\partial}\vec{\hat{D}})^{-1}(-\partial^2)(-\vec{\partial}\vec{\hat{D}})^{-1}|b,x'\rangle \mathcal{J}[A^\perp]\rho^b(x')\,. \end{aligned} \quad (1)$$

Here $g$ is the Yang-Mills coupling constant, $\Pi_i^a(x) = \frac{1}{i}\delta/\delta A_i^\perp(x)$ is the canonical momentum operator conjugate to the transversal gauge field $\vec{A}^\perp(x)$, $\hat{D} = \partial + \hat{A}^\perp$ is the covariant derivative in the adjoint representation ($\hat{A}^{ab} = f^{acb}A^c, f^{abc}$ being the structure constant) and $\mathcal{J} = Det(-\vec{\partial}\vec{\hat{D}})$ is the Faddeev-Popov determinant. Given the fact, that the Faddeev-Popov kernel $(-\vec{\partial}\vec{\hat{D}})$ represents the metric tensor in the color space of transversal gauge connections $A_i^\perp$ the first term in the Hamiltonian is the corresponding Laplace-Beltrami

operator and gives the electric part of the Hamiltonian. The second term gives the magnetic energy with $B_i[A^\perp] = \frac{1}{2}\epsilon_{ijk}[D_j, D_k]$ being the color magnetic field. This term represents a potential for the transversal gauge field. Finally the last term is the so-called Coulomb term, where $\rho^a = \hat{A}_i^{\perp\, ab}\Pi_i^b + \rho_{ext}^a$ is the non-Abelian color charge density, which contains besides the gluonic part also a contribution from external quarks $\rho_{ext}$.

In this letter we solve the Yang-Mills Schrödinger equation $H\Psi = E\Psi$ for the vacuum sector by the variational principle using the following ansatz for the Yang-Mills vacuum wave functional

$$\Psi[A^\perp] = \frac{\mathcal{N}}{\sqrt{\mathcal{J}[A^\perp]}}\, e^{-\frac{1}{2g^2}\int d^3x d^3x' A^{\perp}{}_i^a(x)\omega(x,x')A^{\perp}{}_i^a(x')} \quad (2)$$

where $\mathcal{N}$ is a normalization constant and the kernel $\omega(x, x')$ is determined by minimizing the energy. The wave functional (2) is strongly peaked at the Gribov horizon, where the Faddeev-Popov determinant vanishes and thus reflects the fact, that the dominant infrared configurations, like center vortices or magnetic monopoles, lie on the Gribov horizon [6]. Furthermore, the wave functional (2), being divergent on the Gribov horizon, identifies all configurations on the Gribov horizon, in particular those which are gauge copies of the same orbit. This identification is absolutely necessary to preserve gauge invariance. In addition it topologically compactifies the (first) Gribov region. Thus the pre-exponential factor $\mathcal{J}^{-\frac{1}{2}}[A]$ drastically changes the properties of the vacuum state compared to those of a pure Gaussian, which was used in refs.[7], [8]. In the case of QED, where the Faddeev-Popov determinant becomes a constant eq.(2) represents the exact vacuum wave functional with $\omega(k) = \sqrt{\vec{k}^2}$ being the energy of a free photon with 3-momentum $\vec{k}$.

*3. Schwinger-Dyson equations.* In the evaluation of the vacuum energy $\langle\Psi|H|\Psi\rangle = \int DA^\perp \mathcal{J}[A^\perp]\Psi^*[A^\perp]H\Psi[A^\perp]$ the following ingredients are required:

1. The ghost propagator in the vacuum, which is defined by

$$G = \langle\Psi|(-\vec{\partial}\vec{\hat{D}})^{-1}|\Psi\rangle = (-\vec{\partial}^2)^{-1}\frac{d}{g}\ , \quad (3)$$

where $d$ denotes the ghost form factor, which measures the deviations of the ghost from a free massless field. Evaluating this expectation value in the so-called rainbow ladder approximation, where the self-energy of the ghost is given by the diagram shown in fig.1a one obtains the following integral equation for the ghost form factor $d$ in momentum space

$$\frac{1}{d(k)} = \frac{1}{g} - I_d(k)\ , \quad (4)$$

where ($N_C$ number of colors)

$$I_d(k) = \frac{N_C}{2}\int \frac{d^3q}{(2\pi)^3}\left(1 - (\hat{k}\hat{q})^2\right)\frac{d(k-q)}{(k-q)^2\omega(q)}\ . \quad (5)$$

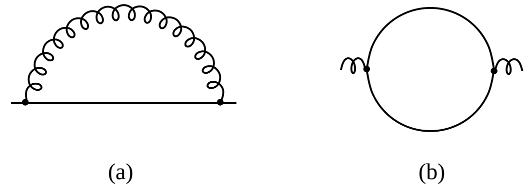


FIG. 1: (a) The ghost self-energy in the rainbow ladder approximation. The full (wavy) line represents the full ghost (gluon) propagator. (b) Ghost loop contribution to the gluon self-energy.

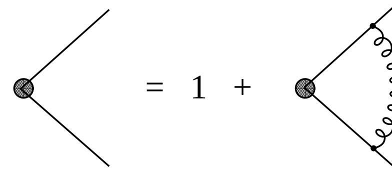


FIG. 2: Diagrammatic illustration of the integral equation for the Coulomb form factor $f$ represented by the fat dot with two entries for external legs.

2. The Coulomb form factor $f$ defined by

$$\langle\Psi|(-\vec{\partial}\vec{\hat{D}})^{-1}(-\Delta)(-\vec{\partial}\vec{\hat{D}})^{-1}|\Psi\rangle := G(-\Delta)fG\ . \quad (6)$$

The calculation of this form factor consistent with the evaluation of the ghost form factor yields the following integral equation

$$f(k) = 1 + I_f(k)\ , \quad (7)$$

$$I_f(k) = \frac{N_C}{2}\int d^3q\left(1 - (\hat{k}\hat{q})^2\right)\frac{d^2(k-q)f(k-q)}{(k-q)^2\omega(q)}\ , \quad (8)$$

which is diagrammatically illustrated in fig.2. In this equation we will replace the full ghost form factor by its bare value $d(k) = 1$, in order to carry out the calculations consistently to 1-loop order.

3. The scalar "curvature" of color orbit space

$$\chi(x, y) = -\frac{1}{4}\frac{1}{N_C^2 - 1}\langle\Psi|\frac{\delta^2 \ln \mathcal{J}[A^\perp]}{\delta A^{\perp}{}_i^a(x)\delta A^{\perp}{}_i^a(y)}|\Psi\rangle\ , \quad (9)$$

which is the ghost loop part of the gluon self-energy and represents the ghost part of the color dielectric susceptibility of the Yang-Mills vacuum. To 1-loop order this curvature is given by the diagram shown in fig.1b and equals

$$\begin{aligned} \chi(k) &= I_\chi(k)\ , \\ I_\chi(k) &= \frac{N_C}{4}\int \frac{d^3q}{(2\pi)^3}\left(1 - (\hat{k}\hat{q})^2\right)\frac{d(k-q)d(q)}{(k-q)^2}\ . \end{aligned} \quad (10)$$

Evaluating the expectation value of the Hamiltonian (1) with the trial wave functional (2) to 1-loop order variation of the energy leads to the following gap equation for the kernel $\omega(x, x')$ in the wave function (2)

$$\omega^2(k) = k^2 + \chi^2(k) + I_\omega^{(2)}(k) + 2\chi(k)I_\omega^{(1)}(k) + I_\omega^0\ , \quad (11)$$

where $I^0_\omega$ is an irrelevant constant, which drops out after renormalization, and

$$I^{(n)}_\omega(k) = \frac{N_C}{4}\int\frac{d^3q}{(2\pi)^3}\left(1+(\hat{k}\hat{q})^2\right)\cdot\frac{d(k-q)^2 f(k-q)}{(k-q)^2} \cdot\frac{(\omega(q)-\chi(q))^n-(\omega(k)-\chi(k))^n}{\omega(q)} \,. \qquad (12)$$

Equations (4), (7), (10) and (11) represent four coupled Schwinger-Dyson type of equations for the ghost form factor $d(k)$, the Coulomb form factor $f(k)$, the curvature $\chi(k)$ and the gluon energy $\omega(k)$. These equations contain divergent integrals and require thus regularization and renormalization. Fortunately, the asymptotic infrared and ultraviolet behaviour of the solutions does not depend on the details of the renormalization procedure used.

*4. Asymptotic behaviour.* One can solve the coupled Schwinger-Dyson equations analytically in the ultraviolet $k\to\infty$ in the so-called angular approximation [10]. One finds then the following asymptotic ultraviolet behaviour

$$\omega(k)\to\sqrt{\vec{k}^2}\ ,\ \frac{\chi(k)}{\omega(k)}\to\frac{1}{\sqrt{\ln k/\mu}} \qquad (13)$$

$$d(k)\to\frac{1}{\sqrt{\ln k/\mu}}\ ,\ f(k)\to\frac{1}{\sqrt{\ln k/\mu}}\,, \qquad (14)$$

where $\mu$ is an arbitrary parameter of dimension mass. The first equation means, that the gluons behave asymptotically like free particles, while the second one implies that the space of gauge connections becomes asymptotically flat. The ghost and Coulomb form factors, $d(k)$ and $f(k)$, deviate asymptotically from that of a free massless field by the anomalous dimension factor $1/\sqrt{\ln k/\mu}$.
In the infrared one can rigorously show, that $\chi(k\to 0)=\omega(k\to 0)$ [11]. In addition, adopting the angular approximation [10] one finds the asymptotic solution for $k\to 0$

$$\omega(k)=\chi(k)\sim\frac{1}{k}\ ,\ d(k)\sim\frac{1}{k}\ ,\ f(k\to 0)=1\ . \qquad (15)$$

The first relation implies, that in the infrared the gluon energy diverges and equals its self-energy part generated by the ghost loop, i.e. its free part $\sqrt{\vec{k}^2}$ has dropped out. The infrared diverging gluon energy is a manifestation of gluon confinement. It implies an infrared vanishing gluon propagator, which violates positivity, and accordingly the gluons do not occur as asymptotic particle states in $S$-matrix [12] ,[14]. Furthermore, the infrared diverging ghost form factor and the infrared finite Coulomb form factor $f(k\to 0)=1$ is precisely the infrared behaviour needed to produce a linearly rising confining potential.

*5. Renormalization.* To regularize and renormalize the divergent Schwinger-Dyson equations we use a 3-momentum cut-off and a momentum substraction scheme similar to the one used in refs.[7, 8]. However, in the present case new features arises in the renormalization of the gap equation (11) due to the full inclusion of the curvature (9). The details will be given elsewhere [11].
After renormalization one obtains the following set of Schwinger-Dyson equations

$$\frac{1}{d(k)} = \frac{1}{d(\mu)}-\Delta I_d(k) \qquad (16)$$

$$\chi(k) = \chi(\mu)+\Delta I_\chi(k) \qquad (17)$$

$$f(k) = f(\mu)+\Delta I_f(k) \qquad (18)$$

$$\omega^2(k) = k^2-\mu^2+(\Delta I_\chi(k))^2+\xi\Delta I_\chi(k)+\Delta I^{(2)}_\omega(k) +2[\chi(\mu)+\Delta I_\chi(k)]\Delta I^{(1)}_\omega(k)+\omega^2(\mu), \qquad (19)$$

where $\mu$ is the renormalization scale and we have introduced the abbreviations

$$\Delta I_d(k)=\lim_{\Lambda\to\infty}\left(I_d(k,\Lambda)-I_d(\mu,\Lambda)\right)\ \ \text{etc .} \qquad (20)$$

Furthermore $\omega(\mu), d(\mu), \chi(\mu), f(\mu)$ and $\xi = 2[\chi(\mu)+I^{(1)}_\omega(\mu)]$ are renormalization constants, which were determined as follows: $\omega(\mu)$ is used to fix the energy scale and drops out from the Schwinger-Dyson equations by rewriting the latter in terms of dimensionless quantities. $d(\mu)$ enters only the Schwinger-Dyson equation (16) for the ghost form factor, which does not contain the remaining renormalization constants. The ultraviolet behaviour of $d(k)$ found above in eq.(14) is independent of $d(\mu)$ but the infrared behaviour of $d(k)$ depends crucially on $d(\mu)$. As long as $d(\mu)$ is smaller than some critical value $d_{cr}$, $d(k)$ approaches a (finite) constant for $k\to 0$. At the critical value $d(\mu)=d_{cr}$ the ghost form factor $d(k)$ diverges for $k\to 0$ and above the critical value $d(\mu)>d_{cr}$ no solution for $d(k)$ exists. This critical value is the only value, which produces the infrared diverging ghost form factor found above analytically (15). Furthermore in $D=3$ (which will be considered elsewhere) a self-consistent solution to the coupled Schwinger-Dyson equations exists only for this critical value. Therefore we choose $d(\mu)=d_{cr}$. Fortunately, the self-consistent solution is quite insensitive to the remaining renormalization constants $\chi(\mu)$ and $\xi$, which we have chosen for the definiteness as $\chi(\mu)=0, f(\mu)=1$ and $\xi=0$.

*6. Numerical Results.* The renormalized coupled Schwinger-Dyson equations eq.(16),(17),(18) and (19) are solved by iteration (without resorting to the angular approximation) and the results are shown in figs.3 and 4.

All these numerical results are in full accord with the ultraviolet and infrared behaviour extracted above analytically within the angular approximation. Finally, fig.5 shows the static quark potential which is obtained in the present approach as the expectation value of the last term in the Yang-Mills Hamiltionian, eq.(1), when the color density $\rho^a$ is identified with that of a static quark-antiquark pair. The obtained potential interpolates between a Coulomb potential at small distances and an (almost) linearly rising confinement potential at large distances. Numerically we find, that its Fourier transform

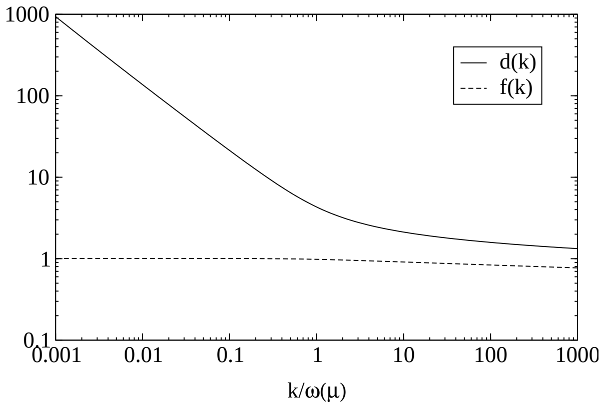


FIG. 3: The ghost form factor $d(k)$ (full line) and the Coulomb form factor $f(k)$ (dashed line).

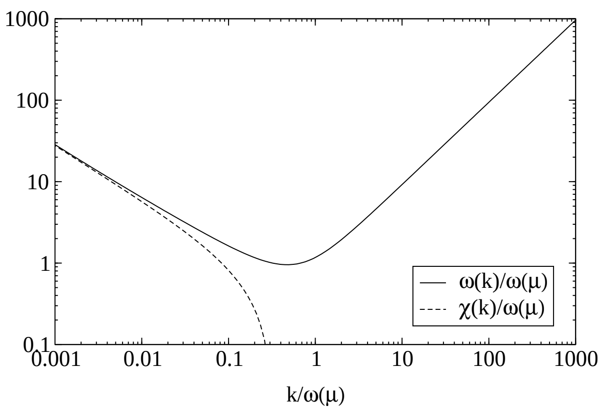


FIG. 4: The gluon energy $\omega(k)$ (full line) and the curvature $\chi(k)$ (dashed line).

diverges for $k \to 0$ as $1/k^{3.7}$ instead of $1/k^4$, which is required for a strictly linear rising potential [15]. Approximating the confining potential by a linear one we can fix the energy scale by the string tension. In ref.[6] it was found that the string tension in the Coulomb potential, $\sigma_{coul}$, is by about a factor of three larger than the asymptotic string tension $\sigma$ extracted from large Wilson loops. Using $\sigma_{coul} = 3\sigma$ and the canonical value $\sigma = (440MeV)^2$ one finds that the minimum in the gluon energy (see fig.4) occurs at $k_{min} \approx 1.4GeV$, which is of the order of the glue ball mass. This corresponds to a minimal single gluon energy $\omega(k_{min}) \approx 3GeV$.

To summarize, by approximately solving the Yang-Mills Schrödinger equation in Coulomb gauge by means of the variational principle using the trial wave functional (2), which embodies the dominance of the field configurations on the Gribov horizon, we have been able to describe simultaneously quark and gluon confinement. A more detailed presentation will be given elsewhere.

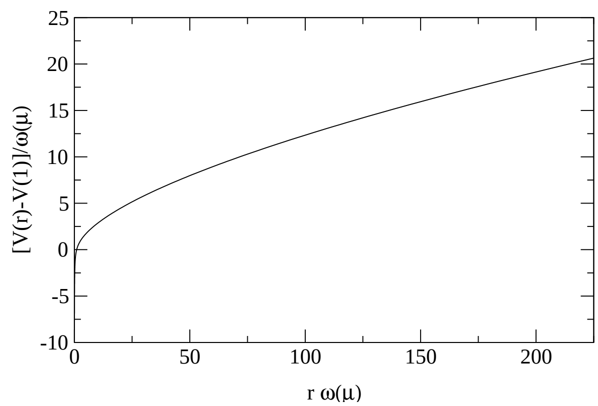


FIG. 5: The static quark potential.

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[1] J. Greensite, Prog. Part. Nucl. Phys. **51** (2003) 1
[2] H. Reinhardt, Nucl. Phys. **B628** (2002) 133
[3] H. Reinhardt, Nucl. Phys.**B503** (1997) 505
[4] V.N. Gribov, Nucl. Phys. **B139** (1978) 1
[5] D. Zwanziger, Nucl. Phys. **B378** (1992) 525
[6] Greensite, Š. Olejnik and D. Zwanziger, hep-lat/0401003
[7] A.P. Szczepaniak and E.S. Swanson, Phys. Rev. D**65** (2002) 025012
[8] A.P. Szczepaniak, hep-ph/0306030
[9] L.v. Smekal, A. Hauck and R. Alkofer, Phys. Rev. Lett **79** (1997) 3591; C.S. Fischer, R. Alkofer and H. Reinhardt, Phys. Rev. **D65** (2002) 094008 and refs. therein
[10] D. Atkinson, J.K. Drohm, P.W. Johnson, K. Stam, J. Math. Phys. **22** (1981)2704
[11] C. Feuchter and H. Reinhardt, to be published
[12] R. Oehme and W. Zimmermann, Phys. Rev. **D21** (1980) 471, ibid. 1661
[13] D. Zwanziger, hep-ph/0312254
[14] An infrared vanishing gluon propagator and an infrared singular ghost propagator are also found in the covariant Schwinger-Dyson approach in Landau gauge [9].
[15] In ref.[13] the power $1/k^{3.6}$ was found.